form $i^*\theta$ is just the arc length form ds as we mentioned above. It is absolutely crucial for the rest of this course to understand the meaning of the form $i^*\Theta_{12}$.

Consider a circle of latitude on a sphere of radius R. To fix the notation, suppose that the circle is at angular distance v from the north pole and that we use u as angular coordinates along the circle. Take the ribbon adapted to the sphere, so e_1 is the unit tangent vector to the circle of latitude and e_2 is the unit tangent vector to the circle of longitude chosen as above. Problem **10** then implies that $i^*\Theta_{12} = -cosvdu$.

12. Let C be a straight line (say a piece of the z-axis) parametrized according to arc length and let e_2 be rotating at a rate $f(s)$ about C (so, for example, $e_2 = \cos f(s)\mathbf{i} + \sin f(s)\mathbf{j}$ where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x and y directions). What is $i^*\Theta_{12}$?

To continue our understanding of Θ_{12}, let us consider what it means for two ribbons, $i : I \to H$ and $j : I \to H$ to have the same value of the pullback of Θ_{12} at some point $s_0 \in I$ (where I is some interval on the real line). So

$$(i^*\Theta_{12})_{|s=s_0} = (j^*\Theta_{12})_{|s=s_0}.$$

There is a (unique) left multiplication, that is a unique Euclidean motion, which carries $i(s_0)$ to $j(s_0)$. Let assume that we have applied this motion so we assume that $i(s_0) = j(s_0)$. Let us write

$$i(s) = (C(s), e_1(s), e_2(s), e_3(s)), \quad j(s) = (D(s), f_1(s), f_2(s).f_3(s))$$

and we are assuming that $C(s_0) = D(s_0)$, $C'(s_0) = e_1(s_0) = f_1(s_0) = D'(s_0)$ so the curves C and D are tangent at s_0, and that $e_2(s_0) = f_2(s_0)$ so that the planes of the ribbon (spanned by the first two orthonormal vectors) coincide. Then our condition about the equality of the pullbacks of Θ_{12} asserts that

$$((e_2' - f_2')(s_0), e_1(s_0)) = 0$$

and of course $((e_2' - f_2')(s_0), e_2(s_0)) = 0$ automatically since $e_2(s)$ and $f_2(s)$ are unit vectors. So the condition is that the relative change of e_2 and f_2 (and similarly e_1 and f_1) at s_0 be normal to the common tangent plane to the ribbon.

2.22 Developing a ribbon.

We will now drop one dimension, and consider ribbons in the plane (or, if you like, ribbons lying in a fixed plane in three dimensional space). So all we have is θ and Θ_{12} . Also, the orientation of the curve and of the plane completely determines e_2 as the unit vector in the plane perpendicular to the curve and such that e_1, e_2 give the correct orientation. so a ribbon in the plane is the same as an oriented curve.

13. Let $k = k(s)$ be any continuous function of s. Show that there is a ribbon in the plane whose base curve is parametrized by arc length and for which

$j^*\Theta_{12} = kds$. Furthermore, show that this planar ribbon (curve) is uniquely determined up to a planar Euclidean motion.

It follows from the preceding exercise, that we have a way of associating a curve in the plane (determined up to a planar Euclidean motion) to any ribbon in space. It consists of rocking and rolling the ribbon along the plane in such a way that infinitesimal change in the e_1 and e_2 are always normal to the plane. Mathematically, it consists in solving problem **13** for the $k = k(s)$ where $i^*\Theta_{12} = kds$ for the ribbon. We call this operation *developing* the ribbon onto a plane. In particular, if we have a curve on a surface, we can consider the ribbon along the curve induced by the surface. In this way, we may talk of developing the surface on a plane along the given curve. Intuitively, if the surface were convex, this amounts to rolling the surface on a plane along the curve.

noindent**14.** What are results of developing the ribbons of Problem **12** and the ribbon we associated to a circle of latitude on the sphere?

2.23 Parallel transport along a ribbon.

Recall that a ribbon is a curve in the space, H, of all Euclidean frames, having the property that the base point, that is the C of the frame (C, e_1, e_2, e_3) has non-vanishing derivative at all points. So C defines a curve in Euclidean three space with nowhere vanishing tangent. We will parameterize this curve (and the ribbon) by arc length. By a unit vector field tangent to the ribbon we will mean a curve, $v(s)$ of unit vectors everywhere tangent to the ribbon, so

$$v(s) = \cos\alpha(s)\ e_1(s) + \sin\alpha(s)\ e_2(s). \tag{2.34}$$

We say that the vector field is *parallel* along the ribbon if the infinitesimal change in v is always normal to the ribbon, i.e. if

$$(v'(s), e_1(s)) \equiv (v'(s), e_2(s)) \equiv 0.$$

Recall the form $\Theta_{12} = kds$ from before.

15. Show that the vector field as given above is parallel if and only if the function α satisfies the differential equation

$$\alpha' + k = 0.$$

Conclude that the notion of parallelism depends only on the form Θ_{12}. Also conclude that given any unit vector, v_0 at some point s_0, there is a unique parallel vector field taking on the value v_0 at s_0. The value $v(s_1)$ at some second point is called the *parallel transport* of v_0 (along the ribbon) from s_0 to s_1.

16. What is the condition on a ribbon that the tangent vector to the curve itself, i.e. the vector field e_1, be parallel? Which circles on the sphere are such that the associated ribbon has this property?

Suppose the ribbon is closed, i.e. $C(s+L) = C(s),\ e_1(s+L) = e_1(s),\ e_2(s+L) = e_2(s)$ for some length L. We can then start with a vector v_0 at point s_0 and transport it all the way around the ribbon until we get back to the same point, i.e. transport from s_0 to $s_0 + L$. The vector v_1 we so obtain will make some angle, call it Φ with the vector v_0. The angle Φ is called the *holonomy* of the (parallel transport of the) ribbon.

17. Show that Φ is independent of the choice of s_0 and v_0. What is its expression in terms of Θ_{12}?

18. What is the holonomy for a circle on the sphere in terms of its latitude.

19. Show that if the ribbon is planar (so e_1 and e_2 lie in a fixed plane) a vector field is parallel if and only if it is parallel in the usual sense of Euclidean geometry (say makes a constant angle with the x-axis). But remember that the curve is turning. So the holonomy of a circle in the plane is $\pm 2\pi$ depending on the orientation. Similarly for the sum of the exterior angles of a triangle (think of the corners as being rounded out).

Convince yourself of the following fact which is not so easy unless you know the trick: Show that for any smooth simple closed curve (i.e. one with no self intersections) in the plane the holonomy is always $\pm 2\pi$.

Exercises **15**,**17**, and **19**, together with the results above give an alternative interpretation of parallel transport: develop the ribbon onto the plane and then just translate the vector v_0 in the Euclidean plane so that its origin lies at the image of s_1. Then consider the corresponding vector field along the ribbon.

The function k in $\Theta_{12} = kds$ is called the geodesic curvature of the ribbon. The integral $\int \Theta_{12} = \int kds$ is called the *total geodesic curvature* of the ribbon. It gives the total change in angle (including multiples of 2π) between the tangents to the initial and final points of the developed curve.

2.24 Surfaces in $\mathbf{R^3}$.

We let M be a two dimensional submanifold of $\mathbf{R}^3$ and $\mathcal{O}$ its bundle of adapted frames. We have a "projection" map

$$\pi : \mathcal{O} \to M, \quad (m, e_1, e_2, e_3) \mapsto m,$$

which we can also write

$$\pi = m.$$

Suppose that we consider the "truncated" version of the adapted bundle of frames $\tilde{\mathcal{O}}$ where we forget about e_3. That is, let consist of all $(m, e_1.e_2)$ where $m \in M$ and e_1, e_2 is an orthonormal basis of the tangent space TM_m to M at m. Notice that the definition we just gave was *intrinsic*. The concept of an orthonormal basis of TM_m depends only on the scalar product on TM_m. The differential of the map $m : \tilde{\mathcal{O}} \to M$ at a point (m, e_1, e_2) sends a tangent vector ξ to $\tilde{\mathcal{O}}$ at (m, e_1, e_2, e_3) to a tangent vector to M at m, and the scalar product of this image vector with e_1 is a linear function of ξ. We have just given an intrinsic of θ_1. (By abuse of language I am using this same letter θ_1 for the form (dm, e_1) on $\tilde{\mathcal{O}}$ as e_3 does not enter into its definition.) Similarly, we see that θ_2 is an intrinsically defined form. From their very definitions, the forms θ_1 and θ_2 are linearly independent at every point of $\tilde{\mathcal{O}}$. Therefore the forms $d\theta_1$ and $d\theta_2$ are intrinsic, and this proves that the form Θ_{12} is intrinsic. Indeed, if we had two linear differential forms σ and τ on $\mathcal{O}$ which satisfied

$$
\begin{aligned}
d\theta_1 &= \sigma \wedge \theta_2, \\
d\theta_1 &= \tau \wedge \theta_2 \\
d\theta_2 &= -\sigma \wedge \theta_1 \\
d\theta_2 &= -\tau \wedge \theta_1
\end{aligned}
$$

then the first two equations give

$$(\sigma - \tau) \wedge \theta_2 \equiv 0$$

which implies that $(\sigma - \tau)$ is a multiple of θ_2 and the last two equations imply that $\sigma - \tau$ is a multiple of θ_1 so $\sigma = \tau$. The next few problems will give a (third) proof of Gauss's theorema egregium. They will show that

$$d\Theta_{12} = -\pi^*(K)\theta_1 \wedge \theta_2$$

where K is the Gaussian curvature.

This assertion is local (in M), so we may temporarily make the assumption that M is orientable - this allows us to look at the sub-bundle $\overline{\mathcal{O}} \subset \mathcal{O}$ of oriented frames, consisting of those frames for which e_1, e_2 form an oriented basis of TM_m and where e_1, e_2, e_3 an oriented frame on $\mathbf{R}^3$.

Let dA denote the (oriented) area form on the surface M. (A bad but standard notation, since we the area form is not the differential of a one form, in general.) Recall that when evaluated on any pair of tangent vectors, η_1, η_2 at $m \in M$ it is the oriented area of the parallelogram spanned by η_1 and η_2, and this is just the determinant of the matrix of scalar products of the η's with any oriented orthonormal basis. Conclude

20. Explain why

$$\pi^* dA = \theta_1 \wedge \theta_2.$$

The third component, e_3 of any frame is completely determined by the point on the surface and the orientation as the unit normal, n to the surface. Now n

can be thought of as a map from M to the unit sphere, S in $\mathbf{R}^3$. Let dS denote the oriented area form of the unit sphere. So n^*dS is a two form on M and we can define the function K by

$$n^*dS = KdA.$$

21 Show that he function K is Gaussian curvature of the surface.

22. Show that

$$n^*dS = \Theta_{31} \wedge \Theta_{32}$$

and

23. Conclude that

$$d\Theta_{12} = -\pi^*(KdA).$$

We are going to want to apply Stokes' theorem to this formula. But in order to do so, we need to integrate over a two dimensional region. So let U be some open subset of M and let

$$\psi : U \to \pi^{-1}U \subset \mathcal{O}$$

be a map satisfying

$$\pi \circ \psi = id.$$

So ψ assigns a frame to each point of U in a differentiable manner. Let C be a curve on M and suppose that C lies in U. Then the surface determines a ribbon along this curve, namely the choice of frames from which e_1 is tangent to the curve (and pointing in the positive direction). So we have a map $R : C \to \mathcal{O}$ coming from the geometry of the surface, and (with now necessarily different notation from the preceding section) $R^*\Theta_{12} = kds$ is the geodesic curvature of the ribbon as studied above. Since the ribbon is determined by the curve (as M is fixed) we can call it the geodesic curvature of the curve. On the other hand, we can consider the form $\psi^*\Theta_{12}$ pulled back to the curve. Let

$$\psi \circ C\ (s) = (C(s), f_1(s), f_2(s), n(s))$$

and let $\phi(s)$ be the angle that $e_1(s)$ makes with $f_1(s)$ so

$$e_1(s) = \cos\phi(s) f_1(s) + \sin\phi(s) f_2(s), \quad e_2(s) = -\sin\phi(s) f_1(s) + \cos\phi(s) f_2(s).$$

24. Let $C^*\psi^*\Theta_{12}$ denote the pullback of $\psi^*\Theta_{12}$ to the curve. Show that

$$kds = d\phi + C^*\psi^*\Theta_{12}.$$

Conclude that

Proposition 2 *The total geodesic curvature* $= \phi(b) - \phi(a) + \int_C \psi * \Theta_{12}$ *where* $\phi(b) - \phi(a)$ *denotes the total change of angle around the curve.*

How can we construct a ψ? Here is one way that we described earlier: Suppose that U is a coordinate chart and that x_1, x_2 are coordinates on this chart. Then $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ are linearly independent vectors at each point and we can apply Gram Schmidt to orthonomalize them. This give a ψ and the angle ϕ above is just the angle that the vector e_1 makes with the $x-$axis in this coordinate system. Suppose we take C to be the boundary of some nice region, D, in U. For example, suppose that C is a triangle or some other polygon with its edges rounded to make a smooth curve. Then the total change in angle is 2π and so

25. Conclude that for such a curve

$$\int\int_D KdA + \int_C kds = 2\pi.$$

The integral of KdA is called the total Gaussian curvature.

26. Show that as the curve actually approaches the polygon, the contribution from the rounded corners approaches the exterior angle of the polygon. Conclude that if a region in a coordinate neighborhood on the surface is bounded by continuous piecewise differentiable arcs making exterior angles at the corners

Proposition 3 *the total Gaussian curvature* + $\sum$ *total geodesic curvatures* + $\sum$ *exterior angles* $= 2\pi$.

27. Suppose that we have subdivided a compact surface into polygonal regions, each contained in a coordinate neighborhood, with f faces, e edges, and v vertices. Let $\xi = f - e + v$. show that

$$\int_M KdA = 2\pi\xi.$$

Chapter 3

Levi-Civita Connections.

3.1 Definition of a linear connection on the tangent bundle.

A **linear connection** ∇ on a manifold M is a rule which assigns a vector field $\nabla_X Y$ to each pair of vector fields X and Y which is bilinear (over $\mathbf{R}$) subject to the rules

$$\nabla_{fX} Y = f \nabla_X Y \tag{3.1}$$

and

$$\nabla_X (gY) = (Xg)Y + g(\nabla_X Y). \tag{3.2}$$

While condition (3.2) is the same as the corresponding condition

$$L_X(gY) = [X, gY] = (Xg)Y + gL_X Y$$

for Lie derivatives, condition (3.1) is quite different from the corresponding formula

$$L_{fX} Y = [fX, Y] = -(Yf)X + fL_X Y$$

for Lie derivatives. In contrast to the Lie derivative, condition (3.1) implies that the value of $\nabla_X Y$ at $x \in M$ depends only on the value $X(x)$.

If $\xi \in TM_x$ is a tangent vector at $x \in M$, and Y is a vector field defined in some neighborhood of x we use the notation

$$\nabla_\xi Y := (\nabla_X Y)(x), \quad \text{where } X(x) = \xi. \tag{3.3}$$

By the preceding comments, this does not depend on how we choose to extend ξ to X so long as $X(x) = \xi$.

While the Lie derivative is an intrinsic notion depending only on the differentiable structure, a connection is an additional piece of geometric structure.

3.2 Christoffel symbols.

These give the expression of a connection in local coordinates: Let $x^1, \ldots, x^n$ be a coordinate system, and let us write

$$\partial_i := \frac{\partial}{\partial x^i}$$

for the corresponding vector fields. Then

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$$

where the functions Γ^k_{ij} are called the **Christoffel symbols**. We will frequently use the shortened notation

$$\nabla_i := \nabla_{\partial_i}.$$

So the definition of the Christoffel symbols is written as

$$\nabla_i \partial_j = \sum_k \Gamma^k_{ij} \partial_k. \tag{3.4}$$

If

$$Y = \sum_j Y^j \partial_j$$

is the local expression of a general vector field Y then (3.2) implies that

$$\nabla_i Y = \sum_k \left\{ \frac{\partial Y^k}{\partial x^i} + \sum_j \Gamma^k_{ij} Y^j \right\} \partial_k. \tag{3.5}$$

3.3 Parallel transport.

Let $C : I \to M$ be a smooth map of an interval I into M. We refer to C as a parameterized curve. We will say that this curve is non-singular if $C'(t) \neq 0$ for any t where $C'(t)$ denotes the tangent vector at $t \in I$. By a **vector field** Z **along** C we mean a rule which smoothly attaches to each $t \in I$ a tangent vector $Z(t)$ to M at $C(t)$. We will let $\mathcal{V}(C)$ denote the set of all smooth vector fields along C. For example, if V is a vector field on M, then the restriction of V to C, i.e. the rule

$$V_C(t) := V(C(t))$$

is a vector field along C. Since the curve C might cross itself, or be closed, it is clear that not every vector field along C is the restriction of a vector field.

On the other hand, if C is non-singular, then the implicit function theorem says that for any $t_0 \in I$ we can find an interval J containing t_0 and a system of coordinates about $C(t_0)$ in M such that in terms of these coordinates the curve is given by

$$x^1(t) = t,\ x^i(t) = 0,\ i > 1$$

for $t \in J$. If Z is a smooth vector field along C then for $t \in J$ we may write

$$Z(t) = \sum_j Z^j(t)\partial_j(t, 0, \dots, 0).$$

We may then define the vector field Y on this coordinate neighborhood by

$$Y(x^1, \dots, x^n) = \sum_j Z^j(x^1)\partial_j$$

and it is clear that Z is the restriction of Y to C on J. In other words, *locally*, every vector field along a non-singular curve *is* the restriction of a vector field of M. If $Z = Y_C$ is the restriction of a vector field Y to C we can define its "derivative" Z', also a vector field along C by

$$Y_C'(t) := \nabla_{C'(t)}Y. \tag{3.6}$$

If g is a smooth function defined in a neighborhood of the image of C, and h is the pull back of g to I via C, so

$$h(t) = g(C(t))$$

then the chain rule says that

$$h'(t) = \frac{d}{dt}g(C(t)) = C'(t)g,$$

the derivative of g with respect to the tangent vector $C'(t)$. Then if

$$Z = Y_C$$

for some vector field Y on M (and $h = g(C(t))$) equation (3.2) implies that

$$(hZ)' = h'Z + hZ'. \tag{3.7}$$

We claim that there is a unique linear map $Z \mapsto Z'$ defined on all of $\mathcal{V}(C)$ such that (3.7) and (3.6) hold. Indeed, to prove uniqueness, it is enough to prove uniqueness in a coordinate neighborhood, where

$$Z(t) = \sum_j Z^j(t)(\partial_i)_C.$$

Equations (3.7) and (3.6) then imply that

$$Z'(t) = \sum_j \left(Z^{j'}(t)(\partial_j)_C + Z^j(t)\nabla_{C'(t)}\partial_j \right). \tag{3.8}$$

In other words, any notion of "derivative along C" satisfying (3.7) and (3.6) must be given by (3.8) in any coordinate system. This proves the uniqueness. On the other hand, it is immediate to check that (3.8) satisfies (3.7) and (3.6) if the curve lies entirely in a coordinate neighborhood. But the uniqueness implies that on the overlap of two neighborhoods the two formulas corresponding to (3.8) must coincide, proving the global existence.

We can make formula (3.8) even more explicit in local coordinates using the Christoffel symbols which tell us that

$$\nabla_{C'(t)}\partial_j = \sum_k \Gamma^k_{ij} \frac{dx^i \circ C}{dt}(\partial_k)_C.$$

Substituting into (3.8) gives

$$Z' = \sum_k \left(\frac{dZ^k}{dt} + \sum_{ij} \Gamma^k_{ij} \frac{dx^i \circ C}{dt} Z^j \right) (\partial_k)_C. \tag{3.9}$$

A vector field Z along C is said to be **parallel** if

$$Z'(t) \equiv 0.$$

Locally this amounts to the Z^i satisfying the system of linear differential equations

$$\frac{dZ^k}{dt} + \sum_{ij} \Gamma^k_{ij} \frac{dx^i \circ C}{dt} Z^j = 0. \tag{3.10}$$

Hence the existence and uniqueness theorem for linear homogeneous differential equations (in particular existence over the entire interval of definition) implies that

Proposition 4 *For any $\zeta \in TM_{C(0)}$ there is a unique parallel vector field Z along C with $Z(0) = \zeta$.*

The rule $t \mapsto C'(t)$ is a vector field along C and hence we can compute its derivative, which we denote by C'' and call the **acceleration** of C. Whereas the notion of tangent vector, C', makes sense on any manifold, the acceleration only makes sense when we are given a connection.

3.4 Geodesics.

A curve with acceleration zero is called a **geodesic**. In local coordinates we substitute $Z^k = x^{k'}$ into (3.10) to obtain the equation for geodesics in local coordinates:

$$\frac{d^2x^k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \tag{3.11}$$

where we have written x^k instead of $x^k \circ C$ in (3.11) to unburden the notation. The existence and uniqueness theorem for ordinary differential equations implies that