Chapter 1

The principal curvatures.

1.1 Volume of a thickened hypersurface

We want to consider the following problem: Let $Y \subset \mathbf{R}^n$ be an oriented hypersurface, so there is a well defined unit normal vector, $\nu(y)$, at each point of Y. Let Y_h denote the set of all points of the form

$$y + t\nu(y), \quad 0 \leq t \leq h.$$

We wish to compute $V_n(Y_h)$ where V_n denotes the $n-$dimensional volume. We will do this computation for small h, see the discussion after the examples.

Examples in three dimensional space.

1. Suppose that Y is a bounded region in a plane, of area A. Clearly

$$V_3(Y_h) = hA$$

in this case.

2. Suppose that Y is a right circular cylinder of radius r and height ℓ with outwardly pointing normal. Then Y_h is the region between the right circular cylinders of height ℓ and radii r and $r + h$ so

$$\begin{aligned} V_3(Y_h) &= \pi\ell[(r+h)^2 - r^2] \\ &= 2\pi\ell r h + \pi\ell h^2 \\ &= hA + h^2 \cdot \frac{1}{2r} \cdot A \\ &= A\left(h + \frac{1}{2} \cdot kh^2\right), \end{aligned}$$

where $A = 2\pi r\ell$ is the area of the cylinder and where $k = 1/r$ is the curvature of the generating circle of the cylinder. For small h, this formula is correct, in fact, whether we choose the normal vector to point out of the cylinder or into the cylinder. Of course, in the inward pointing case, the curvature has the opposite sign, $k = -1/r$.

For inward pointing normals, the formula breaks down when $h > r$, since we get multiple coverage of points in space by points of the form $y + t\nu(y)$.

3. Y is a sphere of radius R with outward normal, so Y_h is a spherical shell, and

$$\begin{aligned} V_3(Y_h) &= \frac{4}{3}\pi[(R+h)^3 - R^3] \\ &= h4\pi R^2 + h^2 4\pi R + h^3\frac{4}{3}\pi \\ &= hA + h^2\frac{1}{R}A + h^3\frac{1}{3R^2}A \\ &= \frac{1}{3}\cdot A\cdot\left[3h + 3\frac{1}{R}\cdot h^2 + \frac{1}{R^2}h^3\right], \end{aligned}$$

where $A = 4\pi R^2$ is the area of the sphere.

Once again, for inward pointing normals we must change the sign of the coefficient of h^2 and the formula thus obtained is only correct for $h \leq \frac{1}{R}$.

So in general, we wish to make the assumption that h is such that the map

$$Y \times [0,h] \to \mathbf{R}^n, \quad (y,t) \mapsto y + t\nu(y)$$

is injective. For Y compact, there always exists an $h_0 > 0$ such that this condition holds for all $h < h_0$. This can be seen to be a consequence of the implicit function theorem. But so not to interrupt the discussion, we will take the injectivity of the map as an hypothesis, for the moment.

In a moment we will define the notion of the various averaged curvatures, $H_1, \ldots, H_{n-1}$, of a hypersurface, and find for the case of the sphere with outward pointing normal, that

$$H_1 = \frac{1}{R}, \quad H_2 = \frac{1}{R^2},$$

while for the case of the cylinder with outward pointing normal that

$$H_1 = \frac{1}{2r}, \quad H_2 = 0,$$

and for the case of the planar region that

$$H_1 = H_2 = 0.$$

We can thus write all three of the above the above formulas as

$$V_3(Y_h) = \frac{1}{3}A\left[3h + 3H_1h^2 + H_2h^3\right].$$

1.2 The Gauss map and the Weingarten map.

In order to state the general formula, we make the following definitions: Let Y be an (immersed) oriented hypersurface. At each $x \in Y$ there is a unique (positive) unit normal vector, and hence a well defined **Gauss map**

$$\nu : Y \to S^{n-1}$$

assigning to each point $x \in Y$ its unit normal vector, $\nu(x)$. Here S^{n-1} denotes the unit sphere, the set of all unit vectors in $\mathbf{R}^n$.

The normal vector, $\nu(x)$ is orthogonal to the tangent space to Y at x. We will denote this tangent space by TY_x. For our present purposes, we can regard TY_x as a subspace of $\mathbf{R}^n$: If $t \mapsto \gamma(t)$ is a differentiable curve lying on the hypersurface Y, (this means that $\gamma(t) \in Y$ for all t) and if $\gamma(0) = x$, then $\gamma'(0)$ belongs to the tangent space TY_x. Conversely, given any vector $v \in TY_x$, we can always find a differentiable curve γ with $\gamma(0) = x,\ \gamma'(0) = v$. So a good way to think of a tangent vector to Y at x is as an "infinitesimal curve" on Y passing through x.

Examples:

1. Suppose that Y is a portion of an $(n-1)$ dimensional linear or affine subspace space sitting in $\mathbf{R}^n$. For example suppose that $Y = \mathbf{R}^{n-1}$ consisting of those points in $\mathbf{R}^n$ whose last coordinate vanishes. Then the tangent space to Y at every point is just this same subspace, and hence the normal vector is a constant. The Gauss map is thus a constant, mapping all of Y onto a single point in S^{n-1}.

2. Suppose that Y is the sphere of radius R (say centered at the origin). The Gauss map carries every point of Y into the corresponding (parallel) point of S^{n-1}. In other words, it is multiplication by $1/R$:
$$\nu(y) = \frac{1}{R}y.$$

3. Suppose that Y is a right circular cylinder in $\mathbf{R}^3$ whose base is the circle of radius r in the x^1, x^2 plane. Then the Gauss map sends Y onto the equator of the unit sphere, S^2, sending a point x into $(1/r)\pi(x)$ where $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ is projection onto the x^1, x^2 plane.

Another good way to think of the tangent space is in terms of a **local parameterization** which means that we are given a map $X : M \mapsto \mathbf{R}^n$ where M is some open subset of $\mathbf{R}^{n-1}$ and such that $X(M)$ is some neighborhood of x in Y. Let $y^1, \dots, y^{n-1}$ be the standard coordinates on $\mathbf{R}^{n-1}$. Part of the requirement that goes into the definition of parameterization is that the map X be **regular**, in the sense that its Jacobian matrix

$$dX := \left(\frac{\partial X}{\partial y^1}, \ \cdots \ , \frac{\partial X}{\partial y^{n-1}}\right)$$

whose columns are the partial derivatives of the map X has rank $n-1$ everywhere. The matrix dX has n rows and $n-1$ columns. The regularity condition amounts to the assertion that for each $z \in M$ the vectors,

$$\frac{\partial X}{\partial y^1}(z), \ \cdots \ , \frac{\partial X}{\partial y^{n-1}}(z)$$

span a subspace of dimension $n-1$. If $x = X(y)$ then the tangent space TY_x is precisely the space spanned by

$$\frac{\partial X}{\partial y^1}(y), \ \cdots \ , \frac{\partial X}{\partial y^{n-1}}(y).$$

Suppose that F is a differentiable map from Y to $\mathbf{R}^m$. We can then define its differential, $dF_x : TY_x \mapsto \mathbf{R}^m$. It is a linear map assigning to each $v \in TY_x$ a value $dF_x(v) \in \mathbf{R}^m$: In terms of the "infinitesimal curve" description, if $v = \gamma'(0)$ then

$$dF_x(v) = \frac{dF \circ \gamma}{dt}(0).$$

(You must check that this does not depend on the choice of representing curve, γ.)

Alternatively, to give a linear map, it is enough to give its value at the elements of a basis. In terms of the basis coming from a parameterization, we have

$$dF_x\left(\frac{\partial X}{\partial y^i}(y)\right) = \frac{\partial F \circ X}{\partial y^i}(y).$$

Here $F \circ X : M \to \mathbf{R}^m$ is the composition of the map F with the map X. You must check that the map dF_x so determined does not depend on the choice of parameterization. Both of these verifications proceed by the chain rule.

One immediate consequence of either characterization is the following important property. Suppose that F takes values in a submanifold $Z \subset \mathbf{R}^m$. Then

$$dF_x : TY_x \to TZ_{F(x)}.$$

Let us apply all this to the Gauss map, ν, which maps Y to the unit sphere, S^{n-1}. Then

$$d\nu_x : TY_x \to TS^{n-1}_{\nu(x)}.$$

But the tangent space to the unit sphere at $\nu(x)$ consists of all vectors perpendicular to $\nu(x)$ and so can be identified with TY_x. We define the **Weingarten map** to be the differential of the Gauss map, regarded as a map from TY_x to itself:

$$W_x := d\nu_x, \qquad W_x : TY_x \to TY_x.$$

The **second fundamental form** is defined to be the bilinear form on TY_x given by

$$II_x(v, w) := (W_x v, w).$$

In the next section we will show, using local coordinates, that this form is symmetric, i.e. that

$$(W_x u, v) = (u, W_x v).$$

This implies, from linear algebra, that W_x is diagonizable with real eigenvalues. These eigenvalues, $k_1 = k_1(x), \cdots, k_{n-1} = k_{n-1}(x)$, of the Weingarten map are called the **principal curvatures** of Y at the point x.

Examples:

1. For a portion of $(n-1)$ space sitting in $\mathbf{R}^n$ the Gauss map is constant so its differential is zero. Hence the Weingarten map and thus all the principal curvatures are zero.

2. For the sphere of radius R the Gauss map consists of multiplication by $1/R$ which is a linear transformation. The differential of a linear transformation is that same transformation (regarded as acting on the tangent spaces). Hence the Weingarten map is $1/R \times \mathrm{id}$ and so all the principal curvatures are equal and are equal to $1/R$.

3. For the cylinder, again the Gauss map is linear, and so the principal curvatures are 0 and $1/r$.

We let H_j denote the jth normalized elementary symmetric functions of the principal curvatures. So

$$\begin{aligned} H_0 &= 1 \\ H_1 &= \frac{1}{n-1}(k_1 + \cdots + k_{n-1}) \\ H_{n-1} &= k_1 \cdot k_2 \cdots k_{n-1} \end{aligned}$$

and, in general,

$$H_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} k_{i_1} \cdots k_{i_j}. \tag{1.1}$$

H_1 is called the **mean curvature** and H_{n-1} is called the **Gaussian curvature**. All the principal curvatures are functions of the point $x \in Y$. For notational simplicity, we will frequently suppress the dependence on x. Then the formula for the volume of the thickened hypersurface (we will call this the "volume formula" for short) is:

$$V_n(Y_h) = \frac{1}{n} \sum_{i=1}^{n} \binom{n}{i} h^i \int_Y H_{i-1} d^{n-1} A \tag{1.2}$$

where $d^{n-1}A$ denotes the ($n-1$ dimensional) volume (area) measure on Y.

A immediate check shows that this gives the answers that we got above for the the plane, the cylinder, and the sphere.

1.3 Proof of the volume formula.

We recall that the Gauss map, ν assigns to each point $x \in Y$ its unit normal vector, and so is a map from Y to the unit sphere, S^{n-1}. The Weingarten map, W_x, is the differential of the Gauss map, $W_x = d\nu_x$, regarded as a map of the tangent space, TY_x to itself. We now describe these maps in terms of a local parameterization of Y. So let $X : M \to \mathbf{R}^n$ be a parameterization of class C^2 of a neighborhood of Y near x, where M is an open subset of $\mathbf{R}^{n-1}$. So $x = X(y),\ y \in M$, say. Let

$$N := \nu \circ X$$

so that $N : M \to S^{n-1}$ is a map of class C^1. The map

$$dX_y : \mathbf{R}^{n-1} \to TY_x$$

gives a **frame** of TY_x. The word "frame" means an isomorphism of our "standard" $(n-1)$-dimensional space, $\mathbf{R}^{n-1}$ with our given $(n-1)$-dimensional space, TY_x. Here we have identified $T(\mathbf{R}^{n-1})_y$ with $\mathbf{R}^{n-1}$, so the frame dX_y gives us a particular isomorphism of $\mathbf{R}^{n-1}$ with TY_x.

Giving a frame of a vector space is the same as giving a basis of that vector space. We will use these two different ways of using the word" frame" interchangeably. Let $e_1, \ldots, e_{n-1}$ denote the standard basis of $\mathbf{R}^{n-1}$, and for X and N, let the subscript i denote the partial derivative with respect to the ith Cartesian coordinate. Thus

$$dX_y(e_i) = X_i(y)$$

for example, and so $X_1(y), \ldots, X_{n-1}(y)$ "is" the frame determined by dX_y (when we regard TY_x as a subspace of $\mathbf{R}^n$). For the sake of notational simplicity we will drop the argument y. Thus we have

$$\begin{aligned} dX(e_i) &= X_i, \\ dN(e_i) &= N_i, \\ \text{and so} & \\ W_x X_i &= N_i. \end{aligned}$$

Recall the definition, $II_x(v, w) = (W_x v, w)$, of the second fundamental form. Let (L_{ij}) denote the matrix of the second fundamental form with respect to the basis $X_1, \ldots X_{n-1}$ of TY_x. So

$$\begin{aligned} L_{ij} &= II_x(X_i, X_j) \\ &= (W_x X_i, X_j) \\ &= (N_i, X_j) \end{aligned}$$

so

$$L_{ij} = -(N, \frac{\partial^2 X}{\partial y_i \partial y_j}), \tag{1.3}$$

the last equality coming from differentiating the identity

$$(N, X_j) \equiv 0$$

in the ith direction. In particular, it follows from (1.3) and the equality of cross derivatives that

$$(W_x X_i, X_j) = (X_i, W_x X_j)$$

and hence, by linearity that

$$(W_x u, v) = (u, W_x v)\ \forall u, v \in TY_x.$$

We have proved that the second fundamental form is symmetric, and hence the Weingarten map is diagonizable with real eigenvalues.

Recall that the principal curvatures are, by definition, the eigenvalues of the Weingarten map. We will let

$$W = (W_{ij})$$

denote the matrix of the Weingarten map with respect to the basis $X_1, \ldots, X_{n-1}$. Explicitly,

$$N_i = \sum_j W_{ji} X_j.$$

If we write $N_1, \ldots, N_{n-1}, X_1, \ldots, X_{n-1}$ as column vectors of length n, we can write the preceding equation as the matrix equation

$$(N_1, \ldots, N_{n-1}) = (X_1, \ldots, X_{n-1})W. \tag{1.4}$$

The matrix multiplication on the right is that of an $n \times (n-1)$ matrix with an $(n-1) \times (n-1)$ matrix. To understand this abbreviated notation, let us write it out in the case $n = 3$, so that X_1, X_2, N_1, N_2 are vectors in $\mathbf{R}^3$:

$$X_1 = \begin{pmatrix} X_{11} \\ X_{12} \\ X_{13} \end{pmatrix},\ X_2 = \begin{pmatrix} X_{21} \\ X_{22} \\ X_{23} \end{pmatrix},\ N_1 = \begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \end{pmatrix},\ N_2 = \begin{pmatrix} N_{21} \\ N_{22} \\ N_{23} \end{pmatrix}.$$

Then (1.4) is the matrix equation

$$\begin{pmatrix} N_{11} & N_{21} \\ N_{12} & N_{22} \\ N_{13} & N_{23} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \\ X_{13} & X_{23} \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}.$$

Matrix multiplication shows that this gives

$$N_1 = W_{11} X_1 + W_{21} X_2, \quad N_2 = W_{12} X_1 + W_{22} X_2,$$

and more generally that (1.4) gives $N_i = \sum_j W_{ji} X_j$ in all dimensions.

Now consider the region Y_h, the thickened hypersurface, introduced in the preceding section except that we replace the full hypersurface Y by the portion $X(M)$. Thus the region in space that we are considering is

$$\{X(y) + \lambda N(y), y \in M\ ,\ 0 < \lambda \le h\}.$$

It is the image of the region $M \times (0,h] \subset \mathbf{R}^{n-1} \times \mathbf{R}$ under the map

$$(y, \lambda) \mapsto X(y) + \lambda N(y).$$

We are assuming that this map is injective. By (1.4), it has Jacobian matrix (differential)

$$J = (X_1 + \lambda N_1, \ldots, X_{n-1} + \lambda N_{n-1}, N) =$$

$$(X_1, \ldots, X_{n-1}, N) \begin{pmatrix} (I_{n-1} + \lambda W) & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.5}$$

The right hand side of (1.5) is now the product of two n by n matrices.

The change of variables formula in several variables says that

$$V_n(h) = \int_M \int_0^h |\det J| dh dy_1 \cdots dy_{n-1}. \tag{1.6}$$

Let us take the determinant of the right hand side of (1.5). The determinant of the matrix $(X_1, \ldots, X_{n-1}, N)$ is just the (oriented) n dimensional volume of the parallelepiped spanned by $X_1, \ldots, X_{n-1}, N$. Since N is of unit length and is perpendicular to the X's, this is the same as the (oriented) $n-1$ dimensional volume of the parallelepiped spanned by $X_1, \ldots, X_{n-1}$. Thus, "by definition",

$$|\det(X_1, \ldots, X_{n-1}, N)| dy_1 \cdots dy_{n-1} = d^{n-1}A. \tag{1.7}$$

(We will come back shortly to discuss why this is the right definition.) The second factor on the right hand side of (1.5) contributes

$$\det(1 + \lambda W) = (1 + \lambda k_1) \cdots (1 + \lambda k_{n-1}).$$

For sufficiently small λ, this expression is positive, so we need not worry about the absolute value sign if h small enough. Integrating with respect to λ from 0 to h gives (1.2).

We proved (1.2) *if* we define $d^{n-1}A$ to be given by (1.7). But then it follows from (1.2) that

$$\frac{d}{dh} V_n(Y_h)_{|h=0} = \int_Y d^{n-1}A. \tag{1.8}$$

A moment's thought shows that the left hand side of (1.8) is exactly what we want to mean by "area": it is the "volume of an infinitesimally thickened region". This justifies taking (1.7) as a definition. Furthermore, although the definition (1.7) is only valid in a coordinate neighborhood, and seems to depend on the choice of local coordinates, equation (1.8) shows that it is independent of the local description by coordinates, and hence is a well defined object on Y. The functions H_j have been defined independent of any choice of local coordinates. Hence (1.2) works globally: To compute the right hand side of (1.2) we may have to break Y up into patches, and do the integration in each patch, summing the pieces. But we know in advance that the final answer is independent of how we break Y up or which local coordinates we use.

1.4 Gauss's theorema egregium.

Suppose we consider the two sided region about the surface, that is

$$V_n(Y_h^+) + V_n(Y_h^-)$$

corresponding to the two different choices of normals. When we replace $\nu(x)$ by $-\nu(x)$ at each point, the Gauss map ν is replaced by $-\nu$, and hence the Weingarten maps W_x are also replaced by their negatives. The principal curvatures change sign. Hence, in the above sum the coefficients of the even powers of h cancel, since they are given in terms of products of the principal curvatures with an odd number of factors. For $n = 3$ we are left with a sum of two terms, the coefficient of h which is the area, and the coefficient of h^3 which is the integral of the Gaussian curvature. It was the remarkable discovery of Gauss that this curvature depends only on the intrinsic geometry of the surface, and not on how the surface is embedded into three space. Thus, for both the cylinder and the plane the cubic terms vanish, because (locally) the cylinder is isometric to the plane. We can wrap the plane around the cylinder without stretching or tearing.

It was this fundamental observation of Gauss that led Riemann to investigate the intrinsic metric geometry of higher dimensional space, eventually leading to Einstein's general relativity which derives the gravitational force from the curvature of space time. A first objective will be to understand this major theorem of Gauss.

An important generalization of Gauss's result was proved by Hermann Weyl in 1939. He showed: if Y is any k dimensional submanifold of n dimensional space (so for $k = 1,\ n = 3$ Y is a curve in three space), let $Y(h)$ denote the "tube" around Y of radius h, the set of all points at distance h from Y. Then, for small h, $V_n(Y(h))$ is a polynomial in h whose coefficients are integrals over Y of intrinsic expressions, depending only on the notion of distance within Y.

Let us multiply both sides of (1.4) on the left by the matrix $(X_1, \dots, X_{n-1})^T$ to obtain

$$L = QW$$

where $L_{ij} = (X_i, N_j)$ as before, and

$$Q = (Q_{ij}) := (X_i, X_j)$$

is called the matrix of the **first fundamental form** relative to our choice of local coordinates. All three matrices in this equality are of size $(n-1)\times(n-1)$. If we take the determinant of the equation $L = QW$ we obtain

$$\det W = \frac{\det L}{\det Q}, \tag{1.9}$$

an expression for the determinant of the Weingarten map (a geometrical property of the embedded surface) as the quotient of two local expressions. For the case $n - 1 = 2$, we thus obtain a local expression for the Gaussian curvature, $K = \det W$.

The first fundamental form encodes the intrinsic geometry of the hypersurface in terms of local coordinates: it gives the Euclidean geometry of the tangent space in terms of the basis $X_1, \ldots, X_{n-1}$. If we describe a curve $t \mapsto \gamma(t)$ on the surface in terms of the coordinates $y^1, \ldots, y^{n-1}$ by giving the functions $t \mapsto y^j(t),\ \jmath = 1, \ldots, n-1$ then the chain rule says that

$$\gamma'(t) = \sum_{j=1}^{n-1} X_j(y(t)) \frac{dy^j}{dt}(t)$$

where

$$y(t) = (y^1(t), \ldots, y^{n-1}(t)).$$

Therefore the (Euclidean) square length of the tangent vector $\gamma'(t)$ is

$$\|\gamma'(t)\|^2 = \sum_{i,j=1}^{n-1} Q_{ij}(y(t)) \frac{dy^i}{dt}(t) \frac{dy^j}{dt}(t).$$

Thus the length of the curve γ given by

$$\int \|\gamma'(t)\| dt$$

can be computed in terms of $y(t)$ as

$$\int \sqrt{\sum_{i,j=1}^{n-1} Q_{ij}(y(t)) \frac{dy^i}{dt}(t) \frac{dy^j}{dt}(t)} \quad dt$$

(so long as the curve lies within the coordinate system).

So two hypersurfaces have the same local intrinsic geometry if they have the same Q in any local coordinate system.

In order to conform with a (somewhat variable) classical literature, we shall make some slight changes in our notation for the case of surfaces in three dimensional space. We will denote our local coordinates by u, v instead of y_1, y_2 and so X_u will replace X_1 and X_v will replace X_2, and we will denote the scalar product of two vectors in three dimensional space by a $\cdot$ instead of $(\ ,\ )$. We write

$$Q = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{1.10}$$

where

$$E := X_u \cdot X_u \tag{1.11}$$

$$F := X_u \cdot X_v \tag{1.12}$$

$$G := X_v \cdot X_v \tag{1.13}$$

so

$$\det Q = EG - F^2. \tag{1.14}$$