Chapter 1

The topology of metric spaces

1.1 Metric spaces

A **metric** for a set X is a function d from $X \times X$ to the non-negative real numbers (which we dente by $\mathbb{R}_{\geq 0}$),

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

such that for all $x, y, z \in X$

1. $d(x, y) = d(y, x)$
2. $d(x, z) \leq d(x, y) + d(y, z)$
3. $d(x, x) = 0$
4. If $d(x, y) = 0$ then $x = y$.

The inequality in 2) is known as the **triangle inequality** since if X is the plane and d the usual notion of distance, it says that the length of an edge of a triangle is at most the sum of the lengths of the two other edges. (In the plane, the inequality is strict unless the three points lie on a line.)

Condition 4) is in many ways inessential, and it is often convenient to drop it, especially for the purposes of some proofs. For example, we might want to consider the decimal expansions $.49999\ldots$ and $.50000\ldots$ as different, but as having zero distance from one another. Or we might want to "identify" these two decimal expansions as representing the same point.

A function d which satisfies only conditions 1) - 3) is called a **pseudo-metric**.

A **metric space** is a pair (X, d) where X is a set and d is a metric on X. Almost always, when d is understood, we engage in the abuse of language and speak of "the metric space X".