1.2 The Gauss map and the Weingarten map.

In order to state the general formula, we make the following definitions: Let Y be an (immersed) oriented hypersurface. At each $x \in Y$ there is a unique (positive) unit normal vector, and hence a well defined **Gauss map**

$$\nu : Y \to S^{n-1}$$

assigning to each point $x \in Y$ its unit normal vector, $\nu(x)$. Here S^{n-1} denotes the unit sphere, the set of all unit vectors in $\mathbf{R}^n$.

The normal vector, $\nu(x)$ is orthogonal to the tangent space to Y at x. We will denote this tangent space by TY_x. For our present purposes, we can regard TY_x as a subspace of $\mathbf{R}^n$: If $t \mapsto \gamma(t)$ is a differentiable curve lying on the hypersurface Y, (this means that $\gamma(t) \in Y$ for all t) and if $\gamma(0) = x$, then $\gamma'(0)$ belongs to the tangent space TY_x. Conversely, given any vector $v \in TY_x$, we can always find a differentiable curve γ with $\gamma(0) = x,\ \gamma'(0) = v$. So a good way to think of a tangent vector to Y at x is as an "infinitesimal curve" on Y passing through x.

Examples:

1. Suppose that Y is a portion of an $(n-1)$ dimensional linear or affine subspace space sitting in $\mathbf{R}^n$. For example suppose that $Y = \mathbf{R}^{n-1}$ consisting of those points in $\mathbf{R}^n$ whose last coordinate vanishes. Then the tangent space to Y at every point is just this same subspace, and hence the normal vector is a constant. The Gauss map is thus a constant, mapping all of Y onto a single point in S^{n-1}.

2. Suppose that Y is the sphere of radius R (say centered at the origin). The Gauss map carries every point of Y into the corresponding (parallel) point of S^{n-1}. In other words, it is multiplication by $1/R$:

$$\nu(y) = \frac{1}{R} y.$$

3. Suppose that Y is a right circular cylinder in $\mathbf{R}^3$ whose base is the circle of radius r in the x^1, x^2 plane. Then the Gauss map sends Y onto the equator of the unit sphere, S^2, sending a point x into $(1/r)\pi(x)$ where $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ is projection onto the x^1, x^2 plane.

Another good way to think of the tangent space is in terms of a **local parameterization** which means that we are given a map $X : M \mapsto \mathbf{R}^n$ where M is some open subset of $\mathbf{R}^{n-1}$ and such that $X(M)$ is some neighborhood of x in Y. Let $y^1, \ldots, y^{n-1}$ be the standard coordinates on $\mathbf{R}^{n-1}$. Part of the requirement that goes into the definition of parameterization is that the map X be **regular**, in the sense that its Jacobian matrix

$$dX := \left(\frac{\partial X}{\partial y^1}, \ \cdots \ , \frac{\partial X}{\partial y^{n-1}} \right)$$